Warp Drive Theory

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Abstract

In this essay, the basic ideas behind a superluminal propulsion system, which is based on directed spacetime distortions and, therefore, is called "warp drive", are introduced. Using the 3+1 formalism, the Alcubierre metric tensor is derived, which satisfies the aspired properties of a warp drive. From an exemplary trip to alpha centauri, it becomes clear that the warp drive causes serious problems, which are discussed in the last part of the essay.

The essay and some additional material is provided online:
`http://www.fiedlschuster.eu/c/physics/warpdrive/`

Contents

1 CD ROM

Additional ressources are provided on the enclosed CD ROM.

1. The essay as a PDF.
2. The bibliography as a PDF, including the hyperlinks to the quoted articles.
3. The quoted articles as PDF.
4. Mathematica scripts.

2 Introduction

Ever since mankind has realised that the stars that appear at the night sky are distant suns like ours, somehow naturally the desire sprouts to travel there.

But it would take 160 thousand years for a typical NASA space shuttle to reach only our nearest neighbour star Proxima Centauri and about $4 \cdot 10^9$ years to cross the galaxy.

Physically we are limited to subluminal speed within special relativity, because it would take an infinite amount of energy only to reach the speed of light.

To overcome this flaw, science fiction has come up with the idea of a warp drive or a hyper drive — some kind of drive that circumvents the usual sense of velocity.

The purpose of Miguel Alcubierre's article *The warp drive: hyper-fast travel within general relativity* [1] was to show that it is possible within the framework of general relativity for a starship to travel with superluminal speed.

This essay purposes to introduce the ideas of this "warp drive" and to discuss some of its major problems, as inter alia shown by Van Den Broeck, Coule and Pfenning. (See section 8.)

But despite all occuring problems, there is no known physical obstacle that prohibits the principal idea of a warp drive. So we still can hope that some day we will be able to travel to the stars of our night sky.

References

[1] Miguel Alcubierre. The warp drive: hyper-fast travel within general relativity. *Classical and Quantum Gravity*, 11:L73, 1994. `http://arxiv.org/abs/gr-qc/0009013`.

[2] Wikipedia (de). Space Shuttle. `http://de.wikipedia.org/wiki/Space_Shuttle`.

3 The Possibility of Superluminal Velocity

For distant galaxies we observe cosmological redshifts $z :> 1$, which correspond to velocities greater than the speed of light. This observation can be interpreted like this that these galaxies move away from us with velocities greater than the speed of light.

Since special relativity states that nothing can travel faster than light, this should be confusing. But this velocity does not occur from the movement of the galaxy within space, but from the expansion of space itself. This is a rather vague description. What else should space be than the distribution of massive objects (within it)? But somehow the geometry (which is the property that determines what a distance is) of space and time is such that the distance between two objects increases as time progresses, at least on cosmological length scales.

The Fizeau-Doppler formula

$$1 + z = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \approx 1 + \frac{v}{c} \tag{1}$$

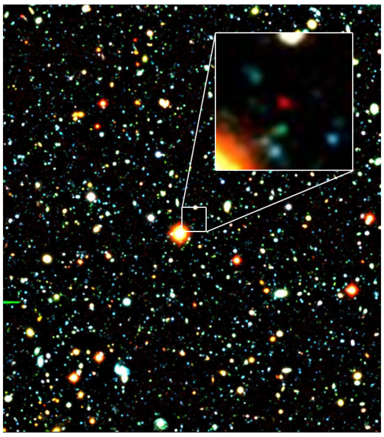
connects the redshift z with the escape speed v. c denotes the speed of light. The very right hand side of the equation shows the Taylor approximation for the non-relativistic case. We do not consider the relativistic formula because the movement is no movement within local (special-relativistic) space, and thus, the objects locally, where the Lorentz transformations would apply, move only with velocities $v \ll c$. Thus, the escape velocity v for cosmological redshifts z is simply

$$v = z \cdot c \ . \tag{2}$$

The galaxy *IOK-1* has a measured redshift of $z = 6.96$, which means an escape velocity of IOK-1 relative to earth that is clealy greater than the speed c of light.

IOK-1 Galaxy

Location of the Galaxy in the Coma Berenices constellation. [2]



The red dot is the galaxy. [1]

Right ascension	13h 23m 59.8s
Declination	$+27^\circ\ 24'\ 56''$
Redshift	6.96
Distance	12.88 GLy

[4]

And, to point it out again, this velocity is not a velocity within space but arises from the expansion of space itself.

Similar to this, we can think of a region of curved space around an object like a starship in a way that space in front of the starship is contracted and space behind the starship is expanded such that the starship appearently moves forward, as shown in the next section.

References

[1] National Astronomical Observatory of Japan. Cosmic archeology uncovers the universe's dark ages. `http://www.subarutelescope.org/Pressrelease/2006/09/13/index.html`, September 2006.

[2] The Stellarium Project. http://www.stellarium.org/.

[3] Wikipedia (en). Cosmological redshift. http://en.wikipedia.org/wiki/Cosmological_redshift.

[4] Wikipedia (en). IOK-1. http://en.wikipedia.org/wiki/IOK-1.

4 The Idea Behind the Warp Drive

Now, as we have seen that the universe appearently allowes superluminal velocities, at least non-locally, by curving space itself, we aim to use this possibility to drive a starship with theoretically arbitrary high speed.

The idea of superluminal speed space travel lies in contracting space in front of the starship, and expanding space behind the starship, such that, for observers outside the disturbed region of space, the starship is travelling with superluminar velocity. [1, p. 1]

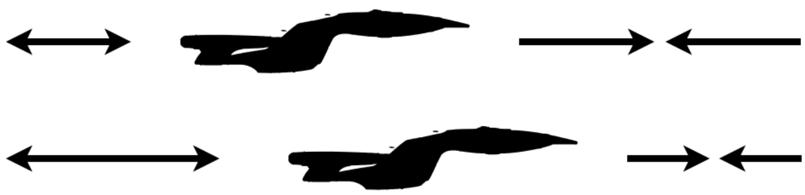


Figure 1: Idea behind the warp drive: Contract the space in front of the starship, expand the space behind it. Thus, the starship moves forward.

The telling name "warp drive" has been introduced in 1966 [2] in the television series *Star Trek*. Alcubierre adopts the same name into science when he says "A propulsion mechanism based on such a local distortion of spacetime just begs to begiven the familiar name of the 'warp drive' of science fiction." [1, p. 8].

Of course, one wants to affect only the starship and its immediate surrounding area with the warp drive, but not the space in a larger distance from the starship. For that reason, one aims to design a kind of distortion bubble around the starship, which will be called "warp bubble".

References

[1] Miguel Alcubierre. The warp drive: hyper-fast travel within general relativity. *Classical and Quantum Gravity*, 11:L73, 1994. http://arxiv.org/abs/gr-qc/0009013.

[2] Startrek.com. Star Trek Episodes. http://www.startrek.com/startrek/view/series/TOS/episodes/index.html.

5 Designing a Warp Bubble

Contents

5.1 Aspired Properties of a Warp Bubble

For the purpose of clarification, let us introduce a kind of coordinate system with three spatial and one temporal coordinates. We want the starship to travel along the x-axis.

The starship is located at the position $(x_s(t), y_s(t), z_s(t))$, where t is the coordinate time parameter. But, since the starship is traveling along the x-axis, we feel free to set $y_s(t) = z_s(t) = 0 \; \forall t$. Thus, the velocity of the starship is $v_s = \frac{\partial x_s(t)}{\partial t}$.

The distance of some spatial point $\boldsymbol{x} := (x, y, z)$ from the starship's centre shall be denoted as $r_s(\boldsymbol{x})$:

$$r_s(\boldsymbol{x}) = \sqrt{(x - x_s)^2 + y^2 + z^2} \tag{3}$$

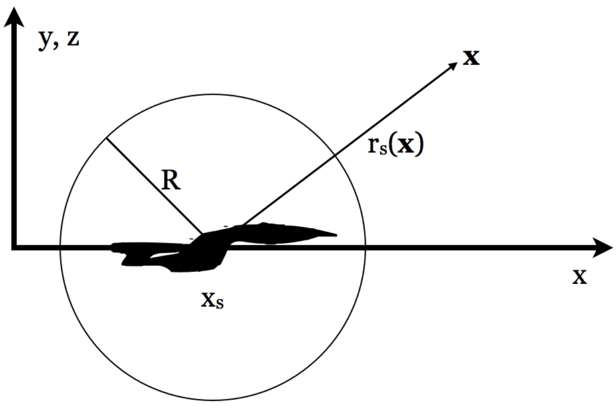


Figure 2: The used coordinate system: The starship moves along the x-axis. R is the radius of the warp bubble.

5.1.1 Making the Ship Move

The primary goal of the warp drive is, of course, to make the starship travel. That means, from the perspective of an outside observer, the starship should move in space as time passes.

But to move must not mean to be translated inside the local region of space around the starship, but to affect space around the starship in a way that moves the whole region of space in relation to the outside space (where the observer is located) in a larger distance from the starship.

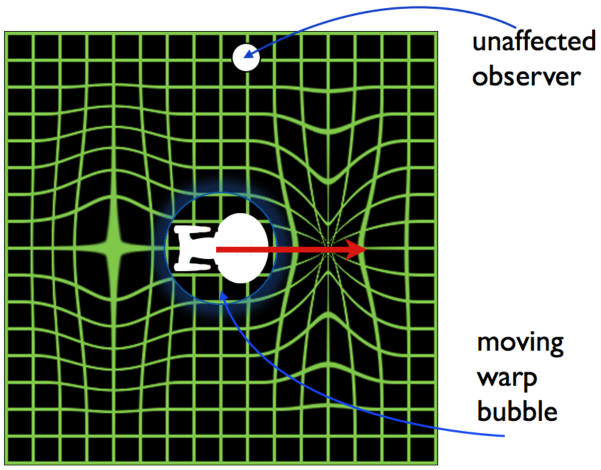


Figure 3: The warp bubble with the ship inside moves forward from the perspective of an observer in certain distance from the warp bubble. [2]

The region of space that is moved in relation to an outside observer, we refer to as "inside the warp bubble".

The term **warp bubble** itself refers to the curved region of space surrounding the starship to be moved using the warp drive.

5.1.2 The Radius of the Warp Bubble

The radius of the warp bubble to be designed shall be denoted as R. By defining a radius, we intend to specify a region where the spatial distortion, i. e. the contraction and expansion, takes place.

The distortion of space shall be confined to a region of the width $2\,\epsilon$ around the radius of the warp bubble:

$$\text{distortions allowed } \forall \boldsymbol{x}: \quad r_s(\boldsymbol{x}) \in [R-\epsilon; R+\epsilon]$$

5.1.3 Normal Space Inside the Warp Bubble

Inside the warp bubble (i. e. $\forall \boldsymbol{x}: \; r_s(\boldsymbol{x}) < R - \epsilon$), there should be "normal spacetime". That means there should be neither spatial nor temporal distortions, but just the usual Minkowski space.

Otherwise the tidal forces may destroy the starship, or temporal effects — like time passing faster in one part of the starship than in another part — would make life on the starship more difficult.

5.1.4 Temporal Synchronism Inside and Outside

One prominent problem of high speeds consists in time dilation effects ("Moving clocks run slow") and the resulting practical problems like seeing the outside world growing old too fast.

Of course, a hypothetical warp drive should avoid these problems. So, ideally, time should pass synchronously inside and outside of the warp bubble.

References

[1] Miguel Alcubierre. The warp drive: hyper-fast travel within general relativity. *Classical and Quantum Gravity*, 11:L73, 1994. `http://arxiv.org/abs/gr-qc/0009013`.

[2] Wikipedia (de). Datei:Star Trek Warp Field.png. `http://de.wikipedia.org/w/index.php?title=Datei:Star_Trek_Warp_Field.png&filetimestamp=20080823041034`.

5.2 How to Describe the Distortion

In general relativity, as in differential geometry, one can describe the curvature properties of the spacetime manifold $\mathcal{M}$ using the metric tensor $g_{\mu\nu}(\boldsymbol{x})$ which is defined for all $\boldsymbol{x} \in \mathcal{M}$.

From this, one can find the other relevant quantities like the **line element** ds, which gives the distance of two infinitesimally near events $\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{dx}$ on the manifold[1].

$$ds^2 = g_{\mu\nu}(\boldsymbol{x})\, dx^\mu\, dx^\nu$$

Given the metric tensor $g_{\mu\nu}(\boldsymbol{x})$, one can find the **Christoffel symbols** $\Gamma^\kappa{}_{\mu\nu}(\boldsymbol{x})$ [2, p. 66]

$$\Gamma^\kappa{}_{\mu\nu} = \frac{1}{2} g^{\kappa\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right)$$

and consequently the **Riemann-Christoffel curvature tensor** $R^d{}_{abc}$, which is a measure for the intrinsic curvature[2] of a manifold [2, p. 158].

$$R^d{}_{abc} = \partial_b \Gamma^d{}_{ac} - \partial_c \Gamma^d{}_{ab} + \Gamma^e{}_{ac} \Gamma^d{}_{eb} - \Gamma^e{}_{ab} \Gamma^d{}_{ec}$$

Furthermore, provided the curvature in terms of the curvature tensor $R^d{}_{abc}$, we can use **Einstein's equations** [2, p. 183] to gain the source of the spacetime distortion, i. e. the matter or energy distribution (given by the **energy-momentum tensor** $T_{\mu\nu}$) we have to create to generate the warp bubble.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -k T_{\mu\nu}$$

[1] Please note: Bold face symbols (like $\boldsymbol{x}$) refer to vectors. This includes elements of higher-dimensional manifolds. In the case of spacetime, the vector is a four-vector. Thus, the bold notation and the index notations are to be regarded as equivalent: $\boldsymbol{x} \equiv (x^\mu)_\mu \equiv x^\mu$, $\mu \in \{0, 1, 2, 3\}$.

[2] The **intrinsic curvature** of a manifold is the curvature that can be detected by the "inhabitants" of this manifold. Contrarily, the **extrinsic curvature** can only be detected by those who have access to the embedding manifold the curved manifold is embedded in. For illustrating examples, see [1, p. 25 ff.].

In the above equation, $R_{\mu\nu} := R^{\rho}{}_{\mu\nu\rho}$ is the Ricci tensor [2, p. 162], $R := g^{\mu\nu} R_{\mu\nu}$ the Ricci scalar, and k some constant, containing the speed of light c and the gravitational constant G: $k = 8\pi G/c^4$.

If the metric tensor $g_{\mu\nu}$ that corresponds to the warp bubble we aim for is provided, we can calculate the ohter quantities of interest. Therefore, we will now begin to look for the metric tensor.

For our description of spacetime, we will use the so-called **3+1 formalism** which describes spacetime as a foliation of spacelike hypersurfaces.

References

[1] Éric Gourgoulhon. 3+1 formalism and bases of numerical relativity. `http://arxiv.org/abs/gr-qc/0703035`, March 2007.

[2] A. N. Lasenby M. P. Hobson, G. Efstathiou. *General Relativity. An Introduction for Physicists.* Cambridge University Press, 2009.

5.3 Foliation of Spacetime

Since we want to compose a propulsion system, which uses deformations in space rather than in time, it is convenient to separate space and time and describe spacetime in a way of foliation where leaves or slices are spacelike hypersurfaces of constant time.

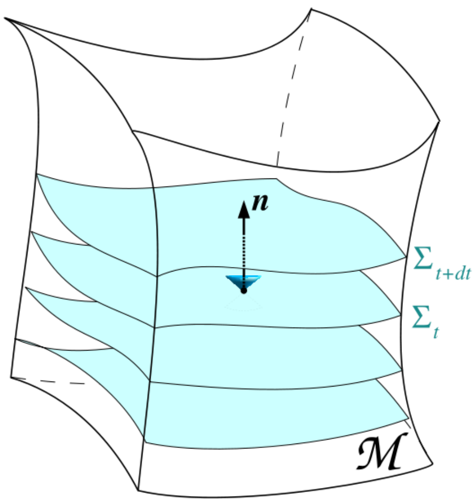


Figure 4: Foliation of the spacetime $\mathcal{M}$ by a family $(\Sigma_t)_{t\in\mathbb{R}}$ of spacelike hypersurfaces Σ_t of constant coordinate time t with normal vector $\boldsymbol{n}$. [3, p. 40]

This so called **3+1 Formalism** is a general approach to general relativity that relies on the slicing of the four-dimensional spacetime by three-dimensional hypersurfaces. [3, p. 11]

5.3.1 Spacetime and Its Leaves

We describe the **spacetime** as a 4-dimensional, real, smooth manifold $\mathcal{M}$ with a Lorentzian metric tensor g with a signature of $(-,+,+,+)$.

Now we foliate the spacetime manifold by a continous set $(\Sigma_t)_{t\in\mathbb{R}}$ of hypersurfaces Σ_t, that covers the manifold $\mathcal{M}$.

$$\mathcal{M} = \bigcup_{t\in\mathbb{R}} \Sigma_t$$

The hypersurfaces are defined as sets of spacetime points with constant coordinate time[3] t.

$$\Sigma_t : \quad \forall p \in \mathcal{M} \quad (\ p \in \Sigma \quad \Leftrightarrow \quad t(p) = t\ )$$

More precisely, we should say that foliation means that there has to exist a smooth scalar field $\hat{t}$ on $\mathcal{M}$ which is regular (i. e. its gradient never vanishes) and allows us to define the hypersurfaces Σ_t as level surfaces of this scalar field:

$$\forall t \in \mathbb{R} \quad \Sigma_t = \big\{ p \in \mathcal{M} :\ \hat{t}(p) = t \big\}$$

But we won't distinguish between t and $\hat{t}$. However, we do note that the hypersurfaces never intersect.

$$\Sigma_t \cap \Sigma_{t'} = \{\} \quad \text{for } t \neq t'$$

Moreover, the slices have to be Cauchy surfaces, i. e. each causal curve (timelike or null) without endpoint intersects each slice Σ once and only once. [3, p. 39]

5.3.2 The Unit Normal Vector

The normal vector $\boldsymbol{n}$ for a point $p \in \mathcal{M}$ is defined to be orthogonal to the slice Σ_t the point p lies in. It can be constructed by using the gradient $\boldsymbol{\nabla}$ of the coordinate time t.

$$\boldsymbol{n} = \lambda\ \boldsymbol{\nabla} t \tag{4}$$

$\boldsymbol{\nabla}$ is the affine connection associated with the metric g of the space time manifold $\mathcal{M}$. Therefore it is called spacetime connection. [3, p. 16]

λ is just a scaling parameter, because we haven't said anything about the length of the normal vector, yet. We take λ such that $\boldsymbol{n}$ is normalised to a length of 1. Therefore we can call $\boldsymbol{n}$ the **unit normal vector**.

$$\boldsymbol{n} = \pm \frac{\boldsymbol{\nabla} t}{\|\boldsymbol{\nabla} t\|} = \pm \frac{1}{\sqrt{-\boldsymbol{\nabla} t \cdot \boldsymbol{\nabla} t}} \boldsymbol{\nabla} t \tag{5}$$

We need the minus sign in the discriminant because the scalar product $\boldsymbol{\nabla} t \cdot \boldsymbol{\nabla} t$ is negative since $\boldsymbol{\nabla} t$ is a timelike vector and the signature of g is $(-,+,+,+)$.

[3] The so-called coordinate time t is the time coordinate we defined for the spacetime manifold $\mathcal{M}$. How it is related to the proper time τ between two events from the perspective of an observer being at these events, we will show in section 5.3.3.

Note that for the same reason, $\boldsymbol{n} \cdot \boldsymbol{n} = -1$, since the considered hypersurfaces Σ_t are spacelike[4].

Note furthermore that we would like to choose $\boldsymbol{n}$ to be the **future-directed** normal vector if t increases towards the future. But since $\boldsymbol{\nabla} t$ is directed into the past (because $\boldsymbol{\nabla} t$ is timelike and we get a minus sign from the metric g), we have to take the minus sign in front of the fraction.

$$\boldsymbol{n} = -\frac{1}{\sqrt{-\boldsymbol{\nabla} t \cdot \boldsymbol{\nabla} t}} \boldsymbol{\nabla} t \tag{6}$$

5.3.3 The Lapse Function

The normalisation factor of the normal vector $\boldsymbol{n}$ in equation (6) (except for the minus sign we used to make $\boldsymbol{n}$ future directed) is called the **lapse function**[5] α. [3, p. 41]

$$\boldsymbol{n} = -\alpha \boldsymbol{\nabla} t, \qquad \alpha = (-\boldsymbol{\nabla} t \cdot \boldsymbol{\nabla} t)^{-1/2} > 0 \tag{7}$$

The hypersurface $\Sigma_{t+\delta t}$ can be obtained[6] from the neighbouring slice Σ_t by the small displacement $\delta t \, \alpha \, \boldsymbol{n}$. Therefore, the vector $\alpha \, \boldsymbol{n}$ is called the **normal evolution vector** [3, p. 42].

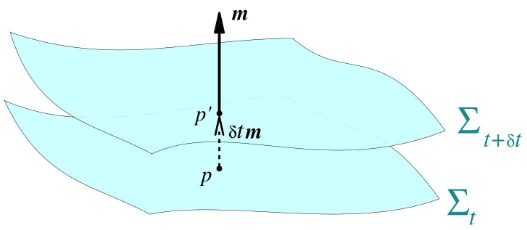


Figure 5: The normal evolution vector $\boldsymbol{m} := \alpha \, \boldsymbol{n}$ [3, p. 41]

To make it precise, let $\boldsymbol{p}$ be a point in one slice Σ_t and $\boldsymbol{p}'$ (spatially) the same point, only a time interval δt later. Then these two points are connected by the small displacement $\delta t \, \alpha \, \boldsymbol{n}$.

$$\boldsymbol{p}' = \boldsymbol{p} + \delta t \, \alpha \, \boldsymbol{n} \tag{8}$$

$$\boldsymbol{p} \in \Sigma_t, \quad \boldsymbol{p}' \in \Sigma_{t+\delta t}, \quad t(\boldsymbol{p}') = t(\boldsymbol{p}) + \delta t$$

So, somehow α states how "dense" the leaves are layed on top of one another.

To make this precise, we follow the path P layed out by the displacements. This path defines the worldline of the observer whose worldline is orthogonal to space leaves Σ_t, the so-called **Eulerian observer** [3, p. 42].

[4] Σ_t : spacelike $\quad \Leftrightarrow \quad \boldsymbol{n}$: timelike

[5] In the ADM formalism [2] and in *3+1 Formalism and Bases of Numerical Relativity* [3], the lapse function is denoted as N: $N \equiv \alpha$. We stick to α because Alcubierre does in his paper [1].

[6] Proof: $p' = p + \delta p, \quad \delta t = \delta p \boldsymbol{\nabla} t, \quad \boldsymbol{n} = -\alpha \boldsymbol{\nabla} t$

$\Rightarrow \quad \delta t = \boldsymbol{\nabla} t \, \delta p \quad \Rightarrow \quad -\alpha \, \delta t = \underbrace{-\alpha \boldsymbol{\nabla} t}_{\boldsymbol{n}} \delta p \quad \Rightarrow \quad -\alpha \, \delta t \, \boldsymbol{n} = -\underbrace{\boldsymbol{n}\boldsymbol{n}}_{-1} \delta p = \delta p$

The interval $\delta\tau$ of proper time generally is given by

$$\delta\tau = \sqrt{-g_{\mu\nu}\, dx^\mu\, dx^\nu}$$

Since we are following the worldline P, the displacement is $dx = dp \equiv \delta p$.

$$\delta\tau = \sqrt{-\delta p_\mu\, \delta p^\mu} = \sqrt{-(\delta t\, \alpha)^2 \underbrace{n_\mu\, n^\mu}_{-1}} = \alpha\, \delta t\ .$$

$$\delta\tau = \alpha\, \delta t \tag{9}$$

Thus, the lapse function α determines the interval of proper time between nearby hypersurfaces as measured by the Eulerian observers [1, p. 3]. This is the reason for its name: it determines the lapse of time.

Note that α is a local quantity, i. e. $\alpha = \alpha(p)$, $p \in \mathcal{M}$. That means that α may stretch or contract time locally. For the warp drive, we wish to accomplish exactly the same thing for space instead of time. The quantity characterising this is the shift vector.

5.3.4 The Shift Vector

As the lapse function α contracts or stretches time, the shift vector we are going to introduce now contracts or stretches space locally.

The shift vector has got this name because it shifts the coordinates x^i of a point $p \in \Sigma_t$ when transiting to the next slice $\Sigma_{t+\delta t}$.

During the last section, when we performed this transition by a small displacement $\delta p := \alpha\, \boldsymbol{n}\, \delta t$, we have assumed that the lines of constant spatial coordinates $(\{p \in \mathcal{M} : x^i(p) := K(\text{some constant})\})$ are orthogonal to the hypersurfaces Σ_t. Therefore, the time-displaced point had the same coordinates in $\Sigma_{t+\delta t}$ as in Σ_t.

Now, we generalise this and allow that the coordinates of a point $\boldsymbol{p} \in \mathcal{M}$ may be locally shifted in space by the shift vector $\boldsymbol{\beta}$ as the coordinate time t varies.

$$\beta : \quad \frac{\partial}{\partial t}\boldsymbol{p} = (\alpha\boldsymbol{n} + \boldsymbol{\beta}), \qquad \beta^0 = 0 \tag{10}$$

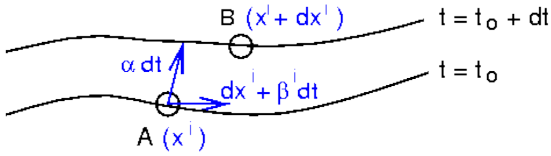


Figure 6: Time evolution: The lapse function α determines the lapse of time. The shift vector β may shift the spatial coordinates. [4]

This generalises the displacement equation (8) to be

$$\boldsymbol{p}' = \boldsymbol{p} + (\alpha\boldsymbol{n} + \boldsymbol{\beta})\delta t\ , \tag{11}$$

or equivalently in tensor notation

$$p'^{\mu} = p^{\mu} + \delta t(\alpha\, n^{\mu} + \beta^{\mu}) \ .$$

Considering the shift vector, the interval $d\tau$ of proper time becomes

$$d\tau = \sqrt{-dp_{\mu}\, dp^{\mu}} = \delta t \sqrt{(\alpha\, \boldsymbol{n} + \boldsymbol{\beta})^2} \ .$$

Note that we have found a way to shift space locally, which will be the core of the warp drive, we are going to look for the metric tensor that incorporates the lapse function and the shift vector.

5.3.5 The metric tensor

In the 3+1 formalism, hypersurfaces Σ_t with a metric tensor $\gamma_{\mu\nu}$ are embedded into the spacetime manifold $\mathcal{M}$ with the metric tensor $g_{\mu\nu}$.

We do know the metric tensor $\gamma_{\mu\nu}$ of the slices Σ_t to satisfy [7]

$$\gamma_{ij} = \delta_{ij} \ , \tag{12}$$

because, as we will see, we demand the slices to be intrinsically flat [1, p. 5].

So, we have to find a relation between both metric tensors, $\gamma_{\mu\nu}$ and $g_{\mu\nu}$, in order to obtain the spacetime manifold's metric $g_{\mu\nu}$, which was our aim in order to calculate other quantities of interest (cf. section 5.2).

Since we already know the normal vectors $\boldsymbol{n}$ for each slice Σ_t, we can use an orthogonal projection operator to relate the metric tensors.

The **orthogonal projection operator** P for a hypersurface Σ_t, projects some vector $\boldsymbol{v} \in \mathcal{M}$ into the hypersurface Σ_t that corresponds to the projection operator. (See figure 7 on page 16.)

$$P: \quad \boldsymbol{v} \mapsto \boldsymbol{v} + (\boldsymbol{n} \cdot \boldsymbol{v})\boldsymbol{n} \tag{13}$$

For the projection, $\boldsymbol{v}$ firstly is projected along the normal vector $\boldsymbol{n}$. Note that the scalar product $(\boldsymbol{n} \cdot \boldsymbol{v})$ produces a minus sign, since n is timelike and the metric's signature is $(-,+,+,+)$. Therefore, the vector $(\boldsymbol{n} \cdot \boldsymbol{v})\boldsymbol{n}$ points in the opposite direction as one might think at first.

Next, the vector along $\boldsymbol{n}$ is added to the original vector, such that the resulting vector $P\boldsymbol{v}$ lies in the hypersurface Σ_t.

$$P\boldsymbol{v} = \boldsymbol{v} + (\boldsymbol{n} \cdot \boldsymbol{v})\boldsymbol{n}$$

In tensor notation, this becomes

$$\begin{aligned} P^{\mu}{}_{\nu} v^{\nu} &= v^{\mu} + n^{\mu} n_{\nu} v^{\nu} \\ &= \left(\delta^{\mu}{}_{\nu} + n^{\mu} n_{\nu}\right) v^{\nu} \end{aligned}$$

[7] Summation indices: $i, j, k \in \{1, 2, 3\}$, $\mu, \nu, \kappa \in \{0, 1, 2, 3\}$.

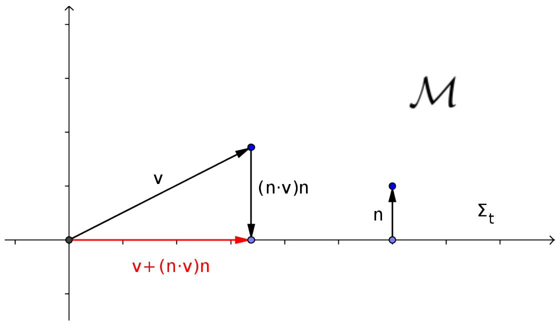


Figure 7: Projection operator: The vector $\boldsymbol{v}$ from the manifold $\mathcal{M}$ is projected into the hypersurface Σ_t. The projected vector is $\boldsymbol{v} + (\boldsymbol{n} \cdot \boldsymbol{v})\boldsymbol{n}$.

or, without the vector v^ν,

$$P^\mu_{\ \nu} = \delta^\mu_{\ \nu} + n^\mu n_\nu$$

We now use this projection operator to project[8] the metric tensor $g_{\mu\nu}$ of the manifold $\mathcal{M}$ into the hypersurface Σ_t.

$$P^\mu_{\ \nu}\, g_{\rho\mu} = \delta^\mu_{\ \nu}\, g_{\rho\nu} + n^\mu n_\nu\, g_{\rho\nu}$$

Summation over the index μ gives the projected metric tensor $\gamma_{\rho\nu}$, which is the metric tensor of the hypersurface Σ_t.

$$\gamma_{\rho\nu} = P^\mu_{\ \nu}\, g_{\rho\mu} = g_{\rho\nu} + n_\rho n_\nu$$

Changing the summation index ρ to μ gives the relation between the metric tensors.

$$g_{\mu\nu} = \gamma_{\mu\nu} - n_\mu\, n_\nu \tag{14}$$

Now we have a way to calculate the metric tensor $g_{\mu\nu}$ of the spacetime manifold $\mathcal{M}$. Setting in the quantities on the right hand side ($n_\mu = (-\alpha, 0, 0, 0)$ [3, eqn. 4.38]), we obtain the metric tensor $g_{\mu\nu}$.

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 & 0 \\ 0 & \gamma_{ij} \end{pmatrix}$$

But note that we haven't taken into account the shift vector $\boldsymbol{\beta}$ so far. Considering the shift vector, the metric tensor becomes [3, p. 58]

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{15}$$

Now that we have the general form of a metric tensor in foliated spacetime, we can look for the metric tensor that describes the warp bubble we want to design.

[8] This is a slightly simplified formulation. To be mathematical exact, we would first have to extend the projection operator to work for dual vectors instead of vectors. For details, see [3, p. 29].

References

[1] Miguel Alcubierre. The warp drive: hyper-fast travel within general relativity. *Classical and Quantum Gravity*, 11:L73, 1994. http://arxiv.org/abs/gr-qc/0009013.

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5.4 The Metric of a Warp Bubble

5.4.1 The Metric Tensor in Foliated Spacetime

As we have seen, the metric tensor $g_{\mu\nu}$ of the spacetime manifold $\mathcal{M}$ can be written in terms of the parameters we used within the foliation description of spacetime. ($\mu, \nu \in \{0,1,2,3\},\ i,j \in \{1,2,3\}$.)

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

α is the lapse function (equation (7)), β_i is the shift vector (equation (10)) and γ_{ij} is the metric tensor of the hypersurfaces (equation (12)).

5.4.2 Finding the Correct Parameters

Now that we have the general form of the metric tensor $g_{\mu\nu}$, we have to find the right parameters α and β_i that characterise the curvature of spacetime and result in the aspired properties of the warp bubble we aim to design (cf. section 5.1 and equations (2) to (5) in [1] and note[9] that $c = G = 1$).

[9] c denotes the speed of light, G the gravitational constant. We will reintroduce these quantities, when we calculate the required energy in section 7.

Metric The distortion effect, i. e. the shift of a certain region of space, shall only occure as time varies. Therefore, if we consider one single slice, one should not see any distortion.

Thus, in order to make the 3-geometry of each slice flat, we set the local spatial metric to be one on the diagonal [1, p. 5].

$$\gamma_{ij} = \delta_{ij} \tag{16}$$

($i, j \in \{1, 2, 3\}$, δ_{ij} is the Kronecker delta.)

Lapse function We do not aim to stretch time somehow. So we can leave the slices in a "constant distance", in other words: set the lapse function to be 1.

$$\alpha = 1 \tag{17}$$

Furthermore, this results in the effect that Euclidean observers are in free fall (because the timelike curves normal to the hypersurfaces are geodesics for $\alpha = 1$ [1, p. 5]).

This does not mean that the whole spacetime is flat. Indeed, this would be contradictory to the spatial shift we want to achieve. But since the spatial shift should be confined to the warp bubble, the rest of spacetime will be essentially flat. [1, p. 5].

Shift vector orthogonal to the direction of motion We don't need the space shifted in the direction orthogonal to the direction in which we want our starship to travel. Therefore, we can set their shift vector components to be zero.

$$\beta^y = \beta^z = 0 \tag{18}$$

Shift vector in the direction of motion We want to travel our starship along the spatial x-axis as it moves through time. Thus, we have to create a curvature, such that the spacetime slices will be shifted along the x-axis.

$$\beta^x = \beta^x(t)$$

As we want to stipulate the velocity $v_s(t) := dx_s(t)/dt$ of our starship, we design a shift that is linear to the ship's velocity [1, eqn. 3].

$$\beta^x = -v_s(t)\, f(r_s(t)) \tag{19}$$

$r_s(t)$ denotes the distance between some space point (x, y, z) and our starship, which is located at the position $(x_s, y_s, z_s) := (x_s(t), 0, 0)$. (See equation (3).)

$$r_s(t) = \sqrt{(x - x_s(t))^2 + y^2 + z^2}$$

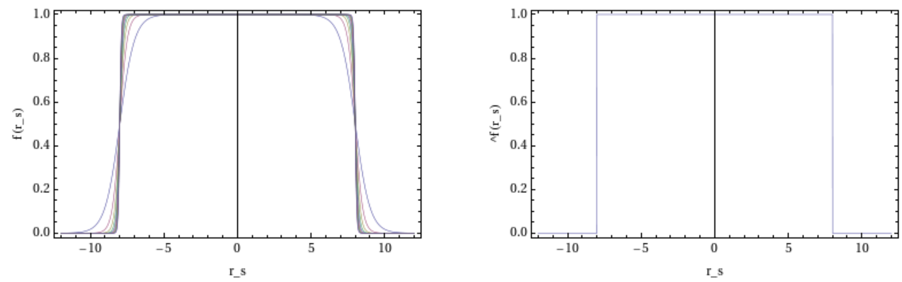
The function $f(r_s(t))$ confines the shift to the spatial region around our starship we refer to as the warp bubble with radius R.

Ideally, the space within the radius R should be x-shifted, resulting in the starship's velocity v_x. Space outside the warp bubble with radius R should not be curved.

$$\beta^x = -v_s(t)\,\hat{f}(r_s(t)), \quad \hat{f}(r_s) := \begin{cases} 1, & r_s \in [-R, R] \\ 0, & r_s > R \end{cases}$$

But such an abrupt transition would be rather unphysical. Thus, we define f to be smooth, but to approach $\hat{f}$ far away from the warp bubble.

$$\beta^x = -v_s(t)\,f(r_s(t)), \quad f(r_s) := \frac{\tanh(\sigma(r_s+R)) - \tanh(\sigma(r_s-R))}{2\,\tanh(\sigma R)} \tag{20}$$



(a) Function $f(r_s)$ for different parameters $\sigma \in \{1, 2, 3, \ldots, 10\}$.

(b) Function $\hat{f}(r_s)$ which is the limit of $f(r_s)$ for $\sigma \to \infty$.

Figure 8: The functions $f, \hat{f}$ limiting the shift function to a certain region of space, the so-called warp bubble.

5.4.3 The Resulting Metric Tensor

Using these parameters we achieved from the required properties of the warp bubble (cf. [1, p. 4]),

$$\begin{aligned} \text{lapse function} \quad & \alpha && = 1 \\ \text{shift vector} \quad & \beta^x && = -v_s(t)\,f(r_s(t)) \\ & \beta^y, \beta^z && = 0 \\ \text{spatial metric} \quad & \gamma_{ij} && = \delta_{ij} \end{aligned}$$

we receive the following expression for the metric tensor $g_{\mu\nu}(\boldsymbol{x})$.

$$\begin{aligned} g_{\mu\nu} &= \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \\ &= \begin{pmatrix} -1 + (v_s(t)\,f(r_s(t)))^2 & -v_s(t)\,f(r_s(t)) & 0 & 0 \\ -v_s(t)\,f(r_s(t)) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{21}$$

5.4.4 The Resulting Line Element

The resulting metric tensor $g_{\mu\nu}$ of the spacetime manifold $\mathcal{M}$ induces the **line element** ds which is the **four-distance** of two nearby events $p, q \in \mathcal{M} : q = p + ds$.

We simply set in the components of $g_{\mu\nu}$ we calculated in equation (21).

$$\begin{aligned} ds^2 &= g_{\mu\nu}\, dx^\mu\, dx^\nu && (22) \\ &= -\left(\alpha^2 - \beta_i \beta^i\right)\, dt^2 + 2\, \beta_i\, dx^i\, dt + \gamma_{ij}\, dx^i\, dx^j \\ &= -dt^2 + \left(dx - v_s\, f(r_s)\, dt\right)^2 + dy^2 + dz^2 && (23) \end{aligned}$$

Here we can see the reason for the minus sign which we have put into the equation (19) for the x-component of the shift vector $\boldsymbol{\beta}$: the x-distance dx in uncurved space becomes $(dx - v_s\, f(r_s)\, dt)$, which means that within the warp bubble (where $f(r_s) = 1$), the x-distance of two events is reduced according to the ship's velocity. If we look at the position of the starship ($x = x_s$), we see that $dx - v_s\, f(r_s)\, dt = 0$ at this position.

This means that, if we look at two spacetime points A and B, where the starship is located at A at a time t_A, and where B is (seen from an outside observer) spatially shifted in the x-direction according to the appearent velocity v_s, and B is a time interval dt later than A, the squared line element becomes

$$\begin{aligned} ds^2\big|_{A,B} &= -dt^2 + (dx - v_s\, f(r_s)\, dt)^2 + dy^2 + dz^2 \\ &= -dt^2 + \left(dx - \frac{dx}{dt}\, dt\right) + 0 + 0 \\ &= -dt^2 \ . && (24) \end{aligned}$$

Thus, the two spacetime points A (where the starship is at the time t_A) and B (where the starship is at the time $t_A + dt$) are **locally** just separated by the passing of time and not by a spatial shift. This means that the starship has not to move locally in order to come from A to B. Whereas an outside observer sees the starship move with the velocity v_s, which is exactly what we want.

Furthermore, we can see from this equation (24) that the coordinate time t passes exactly as fast as the proper time τ of the starship [1, eqn. 13]:

$$dt = d\tau \ ,$$

since $d\tau = -ds^2$ and $ds^2\big|_{A,B} = -dt^2$. This guarantees that the starship's time is synchronous to the time outside the warp bubble, just as we wanted it to be.

5.4.5 The Resulting Curvature

Now that we have the importaint quantities, we would like to visualise the curvature before we concern ourselves with the mass and energy distribution we need to generate the warp bubble.

Since the metric γ_{ij} of the 3-dimensional hypersurfaces is flat, we have to look at the extrinsic curvature, i. e. the way the hypersurfaces are embedded in the spacetime manifold $\mathcal{M}$.

The **extrinsic curvature tensor** $K_{\mu\nu}$ is defined [4] as

$$K_{\mu\nu} = -\frac{1}{2}\,\mathcal{L}_{\boldsymbol{n}} g_{\mu\nu}\ , \tag{25}$$

where $\mathcal{L}_{\boldsymbol{n}}$ denotes the Lie derivative with respect to the normal vector $\boldsymbol{n}$. In the 3+1 formalism, the extrinsic curvature tensor becomes [1, p. 5]

$$K_{ij} = \frac{1}{2\alpha}\left(D_i\beta_j + D_j\beta_i - \frac{\partial g_{ij}}{\partial t}\right)\ ,$$

where D_i denotes the covariant differentiation with respect to the 3-metric γ_{ij}. Setting in α and γ_{ij}, this becomes

$$K_{ij} = \frac{1}{2}\left(\partial_i\,\beta_j + \partial_j\,\beta_i\right)\ . \tag{26}$$

This allowes us to calculate [1, p. 5] the expansion η of the volume elements associated with the Eulerian observers.

$$\begin{aligned} \eta &= -\alpha\,\mathrm{Tr}\,K \\ &= -\alpha\frac{1}{2}\left(\partial_i\,\beta_i + \partial_i\,\beta_i\right) \\ &= v_s\,\frac{x_s}{r_s}\,\frac{df}{dr_s} \end{aligned} \tag{27}$$

The following figure shows a plot of the expansion η against the x-coordinate and the ρ-coordinate, which is a combination of the y- and the z-coordinate: $\rho = \sqrt{y^2 + z^2}$. The plot parameters[10] are $\sigma = 2$, $R = 2$, $v_s = 1$.

In the graph we can see that the warp bubble we have designed indeed does meet our demands: A starship can "sit" in the middle of the bubble where no distortions disturb the ship. In front of the ship (positive x) the volume elements are contracted, behind the ship (negative x) the volume elements are expanded, and therefore, the ship is moving forward from the perspective of an observer outside the warp bubble.

References

[1] Miguel Alcubierre. The warp drive: hyper-fast travel within general relativity. *Classical and Quantum Gravity*, 11:L73, 1994. `http://arxiv.org/abs/gr-qc/0009013`.

[2] Éric Gourgoulhon. 3+1 formalism and bases of numerical relativity. March 2007. `http://arxiv.org/abs/gr-qc/0703035`.

[10] A Mathematica notebook of this plot where the parameters can be varied by using slide controls can be downloaded at `http://demonstrations.wolfram.com/TheAlcubierreWarpDrive/` [3].

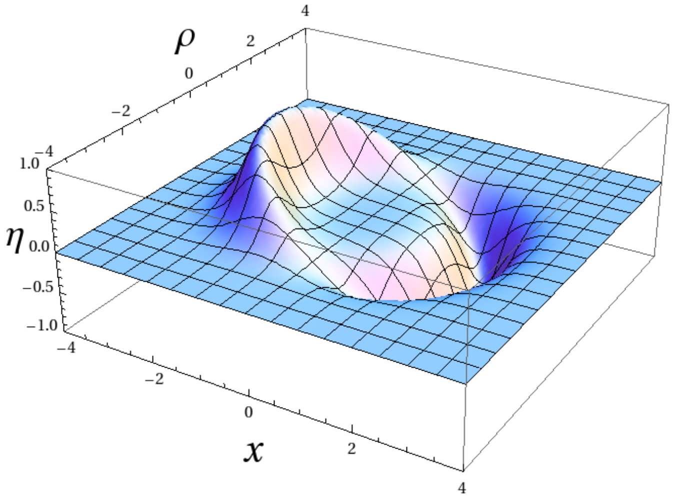


Figure 9: The expansion η of the volume elements associated with the Eulerian observers against the coordinates x and $\rho := \sqrt{y^2 + z^2}$. The parameters are $\sigma = 2$, $R = 2$, $v_s = 1$.

[3] Thomas Mueller. Wolfram Demonstration - The Alcubierre Warp Drive. http://demonstrations.wolfram.com/TheAlcubierreWarpDrive/.

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6 Generation of the Warp Bubble

If we look at **Einstein's equation** we see that every matter or energy distribution (given by the **energy-momentum tensor** $T_{\mu\nu}$) generates curvature in spacetime (given by the **Einstein tensor** $G_{\mu\nu}$).

$$G_{\mu\nu} = k \cdot T_{\mu\nu} \tag{28}$$

k is some constant containing the speed of light c and the gravitational constant G: $k = 8\pi G/c^4$.

Since we know the curvature we would like to generate, we can use Einstein's equation to look for the energy distribution to generate it.

Metric Tensor We calculated the metric tensor in euation (21).

$$g_{\mu\nu} = \begin{pmatrix} -1 + (v_s(t)\, f(r_s(t)))^2 & -v_s(t)\, f(r_s(t)) & 0 & 0 \\ -v_s(t)\, f(r_s(t)) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In this euqation, $v_s(t) := \frac{\partial x}{\partial t}$ is the ships velocity, $r_s(t) := \sqrt{(x - x_s)^2 + y^2 + z^2}$ is the distance from the ship's centre (which is the centre of the warp bubble as well), and $f := \frac{\tanh(\sigma(r_s + R)) - \tanh(\sigma(r_s - R))}{2\,\tanh(\sigma R)}$ the function that defines the shape of the warp bubble, see equation (20).

Christoffel Symbols From that we can calculate the Christoffel symbols $\Gamma^{\kappa}{}_{\mu\nu}(\boldsymbol{x})$ [3, p. 66]

$$\Gamma^{\kappa}{}_{\mu\nu} = \frac{1}{2} g^{\kappa\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right)$$

Riemann Tensor From the Christoffel symbols, we can calculate the Riemann-Christoffel curvature tensor [3, p. 158].

$$R^d{}_{abc} = \partial_b \Gamma^d{}_{ac} - \partial_c \Gamma^d{}_{ab} + \Gamma^e{}_{ac} \Gamma^d{}_{eb} - \Gamma^e{}_{ab} \Gamma^d{}_{ec}$$

By contraction, we get the **Ricci tensor** $R_{\mu\nu}$ and the **Ricci scalar** R [3, p. 162].

$$R_{\mu\nu} = R^\rho{}_{\mu\nu\rho} \quad , \quad R = g^{\mu\nu} R_{\mu\nu}$$

Einstein Tensor And from these quantities, we can calculate the Einstein tensor $G_{\mu\nu}$.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Setting in this into Einstein's equation (28), we obtain the energy-momentum tensor $T_{\mu\nu}$ which represents the energy and matter distribution we are looking for in order to generate a warp bubble.

The evaluation of the energy-momentum tensor proved to be rather extensive. A Mathematica notebook for the calculation is provided on the enclosed CD ROM. For the energy requirements, however, the results of Pfenning [4] are used.

Energy requirements The resulting energy requirements are calculated from the 00-component of the energy-momentum tensor $T_{\mu\nu}$ [4, p. 9].

$$E = \int dx^3 \sqrt{|\det \gamma_{ij}|} \langle T^{00} \rangle \tag{29}$$

In this expressions, γ_{ij} is the metric tensor of the hypersurfaces Σ_t, $\langle T^{00} \rangle$ is the medial energy density of the matter distribution generating the warp bubble.

References

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7 A Spaceflight to Alpha Centauri

To get an impression of the scale of the occuring quantities, we will describe a fictive trip to α-Centauri, the closest star system to earth. α-Centauri is a top candidate for extrasolar life. Thus, the trip could be worth it. [2]

Alpha Centauri's distance D to earth is 4.34 lightyears which is approximately

$$D = 41.06 \cdot 10^{15}\,\mathrm{m}\ .$$

As we will see later, the required amount of energy increases quadratically with the ships velocity v_s during the warp flight. Of course the flight time decreases with increasing speed. So, we feel free to pick just an arbitrary velocity v_s for our trip.

The cruising flight speed of the *Enterprise-D* in *Star Trek* is “Warp 6” which is 392.5c. [6] So, let's take this speed for our trip to alpha centauri.

$$v_s = 392.5\,c = 1.176 \cdot 10^{12}\,\frac{\mathrm{m}}{\mathrm{s}}$$

Flight Time of the One Way Trip The coordinate time T — which is the time that passes on earth as well as in the alpha centauri system — is just

$$T = \frac{D}{v_s} = 9.7\,\text{hours}.$$

Since the warp bubble keeps the proper time τ inside the bubble synchronous to the coordinate time outside, the passed time τ inside the starship is exactly the same:

$$\tau = T = 9.6\,\text{hours}$$

Energy Consumption According to equation (29), the energy E needed to generate the warp bubble is [4, p. 9]

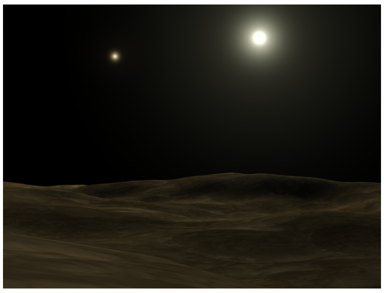
$$\begin{aligned} E &= \int dx^3 \sqrt{|\det \gamma_{ij}|} \langle T^{00} \rangle \\ &= -\frac{v_s^2}{32\pi} \int \frac{\rho^2}{r^2} \left(\frac{df(r)}{dr} \right)^2 dx^3 \\ &= -\frac{1}{12}\, v_s^2 \left(\frac{R^2}{\epsilon} + \frac{2\epsilon}{12} \right) . \end{aligned}$$

In this expressions, γ_{ij} is the metric tensor of the hypersurfaces Σ_t, $\langle T^{00} \rangle$ is the medial energy density of the matter distribution generating the warp bubble, $r := r_s(t = 0)$

α-Centauri

Double Star System

The Centaurus constellation in the southern night sky. The white spot at the bottom is Alpha Centauri. [2]



An artist's rendition of the view from a hypothetical airless planet orbiting Alpha Centauri A [7]

Right ascension	A: 14h 39m 36.5s B: 14h 39m 35.1s
Declination	A: -60° 50′ 02,31″ B: -60° 50′ 13,76″
Spectral type	A: G2V B: K1V
Dst. to earth	4.34 Ly
Age	$4.85 \cdot 10^9$ years
Period	79.9 years
Periastron	11.5 AU
Apastron	36.3 AU

[3], [2]

some time-fixed distance variable from the starships centre (because the total energy is constant), R the radius of the warp bubble and 2ϵ the width of the warp bubble's border.

In order to get SI units, we have to reinstate the speed c of light and the gravitational constant G which have been ignored ($c = G = 1$) before. So we multiply by $1 = \frac{c^2}{G}$ to get energy units on the right hand side.

$$E = -\frac{1}{12\,G}\,(v_s\,c)^2\,\left(\frac{R^2}{\epsilon} + \frac{\epsilon}{12}\right)$$

Please note, that the energy E apparently does not depend on the distance D we want to travel. This might indicate that the warp bubble, once created, is moving forward until it is interrupted by another energy matter distribution. Or it might indicate that we haven't completely understood the properties of the matter distribution we need to generate the warp bubble.

But let us calculate the energy E for some reasonable parameters. The *Enterprise-D* has got a length of about 650 metres. Thus, we take $R = 700\,\mathrm{m}$. The border width 2ϵ is shown to be constrained by Pfenning and Ford [4, eqn. 23]. According to them, due to quantum inequality restrictions, 2ϵ has to be

$$2\epsilon \leq 10^2\,v_s^2\,l_{\mathrm{P}}\ ,$$

where $l_{\mathrm{P}} = \sqrt{\hbar\,G\,/\,c^3} = 1.616252 \cdot 10^{-35}\,\mathrm{m}$ is the Planck length. Thus, if we take the maximum ϵ, we get

$$2\epsilon = 2.2352 \cdot 10^{-09}\,\mathrm{m}\ .$$

Setting in all quantities, the resulting needed energy for the warp bubble is

$$\begin{aligned} E &= -3.4068 \cdot 10^{64}\,\mathrm{J} \\ &= -2.9532 \cdot 10^{22}\,c^2 \text{ masses of the milky way.} \end{aligned} \tag{30}$$

This clearly poses a problem. On the one hand, the modulus energy is enormous, much greater than the total mass of the visible universe, which is about $10^{53}\,\mathrm{kg}$ [5]. On the other hand, the energy is negative — and this is not a matter of convention.

As one can see, as simple the basic ideas and calculations are, the conception of a warp drive proves to cause severe problems. The rest of this essay shall examine some of these problems.

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8 Problems of the Warp Drive

Contents

8.1 Energy Condition Violations

Energy conditions aim to exclude solutions of Einstein's equation regarded as unphysical. But it has to be kept in mind that they are stipulations. If one finds a contradictory result, it is not neccessarily wrong — rather the energy conditions have to be reconsidered.

Weak Energy Condition The weak energy condition stipulates that for every future-pointing timelike vector field $\boldsymbol{x}$, the matter density ρ observed by the corresponding observers is always non-negative [4]:

$$\rho = T_{ab}\, x^a\, x^b \geq 0$$

Since we calculated the total energy requirements to be negative (see equation (30)), the weak energy condition is violated.

Violations of the weak or dominant energy condition can occur in quantum field theory, for example, in the Casimir effect. [3]

Matter sources that violate the weak energy condition are called **exotic**. However, there are limits to how large these violations can be. They are constrained by the so-called **quantum inequalities**. Ford and Pfenning applied these restrictions to the warp drive in their paper *The unphysical nature of "warp drive"* [8] and showed that there is a limitation to the size of the warp bubble's border.We have used this result already in equation (7).

That means, in order to get a warp drive to work, one has either to minimize the amount of neccessary exotic matter, or to find a way to violate the weak energy condition on a greater scale.

Dominant Energy Condition The dominant energy condition stipulates that, in addition to the weak energy condition holding true, for every future-pointing causal vector field (timelike or null) $\boldsymbol{x}$, the vector field $-T^a_{\ b}\, x^b$ must be a future-pointing causal vector, i. e. mass-energy can never be observed to be flowing faster than light [4].

Locally, the starship inside the warp bubble doesn't move faster than light, but as for the matter distribution, which generates the warp bubble, this is not neccessarily true. Indeed, Coule states [3] that one needs to use a matter distribution with tachyonic speed in order to generate a warp bubble.

But in general, the argument is the same as for the weak energy condition: Either one can decrease the neccessary amount of the violation, or one can find a way to perform a greater violation, in order to get the warp drive work.

Strong Energy Condition The strong energy condition stipulates that for every future-pointing timelike vector field $\boldsymbol{x}$, the trace of the tidal tensor measured by the corresponding observers is always non-negative [4]:

$$\left(T_{ab} - \frac{1}{2} T\, g_{ab}\right) x^a\, x^b \geq 0$$

Violating the strong energy condition is easier to justify on physical grounds. Such violations would occur during an inflationary expansion of the universe [3] which is assumed to have happened in an early state of the universe.

And since the warp drive also violates the strong energy condition [1], this gives hope that this violation is not neccessarily an exclusion criterion to the warp drive.

8.2 Energy Requirements

As seen in section 7, the energy requirements $|E|$ for the creation of a warp bubble are enormous: $|E| \approx 3 \cdot 10^{64}$ J for a warp bubble with a radius $R = 700$ m.

Chris Van den Broeck shows in his paper *A 'warp drive' with more reasonable total energy requirements* [2] that a minor modification of the Alcubierre geometry can dramatically improve the total energy requirements for a warp bubble.

The new geometry satisfies the quantum inequality concerning the weak energy condition [2, p. 8] and has the same advantages as the original Alcubierre geometry.

The idea is to keep the surface area of the warp bubble itself microscopically small (seen from the outside), while at the same time expanding the spatial volume inside the bubble, such that a starship can fit into the warp bubble. [2]

Therefore, Broeck extends the Alcubierre line element (eqn. (23))

$$ds^2 = -dt^2 + \left(dx - v_s\, f(r_s)\, dt\right)^2 + dy^2 + dz^2$$

with a factor $B(r_s)$ which expands the spatial volume inside the original Alcubierre warp bubble. Thus, the Broeck line element ds is defined as

$$ds^2 = -dt^2 + B^2(r_s) \left[(dx - v_s(t)\, f(r_s)\, dt)^2 + dy^2 + dz^2\right] \ . \tag{31}$$

In order to create the "pocket" the starship lies in, the weight function $B(r_s)$ should have the following properties.

$$B(r_s) : \begin{cases} = 1 + \lambda, & r_s < \tilde{R}\ , \qquad \lambda : \text{large constant} \\ \in]1; 1+\lambda[, & r_s \in [\tilde{R}; \tilde{R} + \tilde{\Delta}] \\ = 1, & r_s > \tilde{R} + \tilde{\Delta} \end{cases} \tag{32}$$

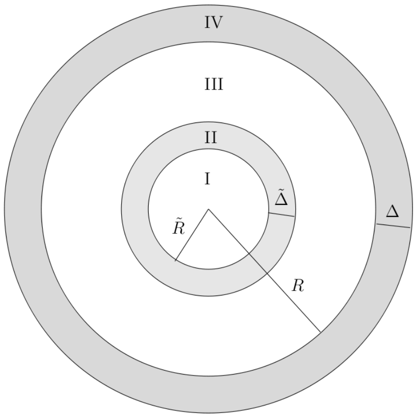


Figure 10: The structure of the Broeck warp bubble: region I, where the starship is located, has got an enlarged volume compared to normal space. II is the transition region from the blown-up part of space to the normal part. In II, the function $B(r_s)$ varies. From region III outward the geometry is the original Alcubierre geometry. Region IV is the wall of the warp bubble. In region IV, f varies. Spacetime is flat, except in the shaded regions II and IV. [2, fig. 1]

To give an example, Broeck chooses some values for the constants

$$\lambda = 10^{17}, \quad \tilde{\Delta} = 10^{-15}\,\mathrm{m}, \quad \tilde{R} = 10^{-15}\,\mathrm{m}, \quad R = 3 \cdot 10^{-15}\,\mathrm{m}$$

and suggests the function $B(r_s)$ to be

$$B = \lambda(-(n-1)\omega^n + n\,\omega^{n-1}) + 1, \quad \omega = \frac{\tilde{R} + \tilde{\Delta} - r_s}{\tilde{\Delta}}, \quad n = 80$$

For these values, which result in a "pocket" for the starship to lie in with an inner diameter of more than 100 metres, the resulting total amount of required energy is in the order of a few solar masses $M_\odot$ which is considerably smaller than the required energy for the original Albicurre warp metric.

$$E \approx -3\,M_\odot$$

One more comment on the weight function $B(r_s)$: the function Broeck suggested ((8.2)) confusingly seems not to fulfil the properties ((32)) we wanted it to have, which becomes clear when looking at the graph of the function.

One can think of an alternative suggestion for B, which has Alcubierre's top hat function $f(r_s)$ as a prototype.

$$B(r_s) = \lambda\,F(\tilde{R}, r_s) + F(R, r_s - 2\tilde{R}) \quad \text{with}$$

$$F(R, r_s) := \frac{\tanh(\sigma(r_s + R)) - \tanh(\sigma(r_s - R))}{2\,\tanh(\sigma\,R)}, \quad \sigma = \frac{10}{\tilde{\Delta}}$$

Nevertheless, Broeck's idea of shrinking the outside surface area of the warp bubble results in considerably more reasonable total energy requirements.

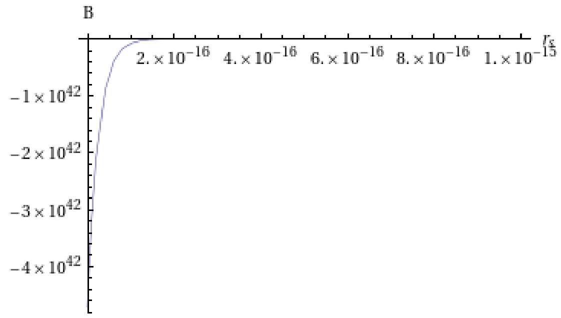
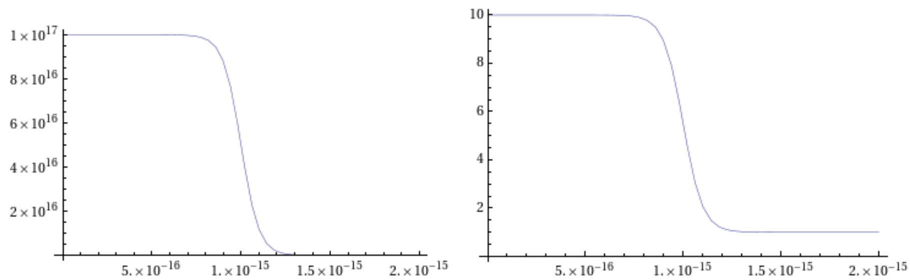


Figure 11: The function B Broeck suggested does not fulfil the aspired properties. It does not fit into the definition (32).



(a) $\lambda = 10^17$

(b) $\lambda = 10$ (Here one can see that B becomes 1 on the right hand side.)

Figure 12: The alternative function B fits into the definition (32).

8.3 You need one to make one?

D. H. Coule argued [3] that one needs to transcend the speed of light in order to construct a warp drive in the first place.

Coule states that in order to make the warp bubble move with speeds greater than the speed of light, the matter distribution creating the bubble has to move with this speed as well.

One possible solution lies in the so-called Krasnikov Tube: one could distribute matter along a track at subluminal velocity and (after this) send a ship along with superluminal speed. [9] But this would mean that the starship would be confined to preset routes rather than stearing at will. Furthermore it would take a long time to create the tubes at subluminal speed.

As an alternative use, Coule suggested [3] that one can use the ideas of the warp drive on very small distances, for example in micro chips which would offer a great performance because the speed of light limit for information transfer would be abolished.

But since we do not fully understand neither how the expansion of the universe is driven, nor how the inflation of the early universe was caused, there is still hope that one can find another way to manipulate the curvature of spacetime, if it turns out that Coule is right.

8.4 Hazardous Matter and Radiation

A warp driven starship may collide with objects in front of the ship during the flight, which would be hazardous to the ship and its crew. Even photons arriving in the front of the ship are blueshifted to very high energies in the region near the border of the warp bubble (which will be called Pfenning region[11]). This hight energy radiation can be lethal to the ship's crew and damage the ship itself. [5]

C. B. Hart et al. showed in their article *On the Problems of Hazardous Matter and Radiation at Faster than Light Speeds in the Warp Drive Space-time* [5] that the Broeck metric we introduced in section 8.2 solves this problem.

The metric was designed such that it has two warped regions. One is the usual Pfenning warped region and the other is the Broeck warped region, which will slow down incoming photons in the neighbourhood of the ship and disrupt and deflect larger objects.

The Broeck metric is, as we introduced it in equation (31),

$$ds^2 = 1 - B^2[dx - v_s\, f(r_s)\, dt]^2 \tag{33}$$

where the function f confines the distortion to the warp bubble as it was in the Alcubierre metric. In contrast to Broeck, Hart defines the weight function $B(r_s)$ to be [5, eqn. 4]

$$B = \left[\frac{1 + \tanh[\sigma(r_s - D)]^2}{2}\right]^{-P}$$

D is the radius of the Broeck warp region. P is a free parameter. The following plot shows B for $\sigma = 3, P = 3, D = 10$.

But note that the weight function $B(r_s)$ Hart suggests does not possess the properties Broeck postulated. See equation (32).

Nevertheless, Hart states that Photons entering the Pfenning region will be accelerated. The Broeck region was designed to slow them down.

The speed of a incoming photon in the distance r_s from the ship as a result [5, eqn. 17] is as follows.

$$v = -v_s\,(1 - f(r_s)) - \frac{1}{B}$$

Again, the plot reveals two problems: Hart states that objects entering the Pfenning region ($r_s = 15$ in figure 14) are accelerated. But equation (8.4) shows that the speed is reduced already in the Pfenning region. Next, the

[11] The region near the border of the warp bubble, i. e. in a distance R from the starship's centre, will be called Pfenning region to distinguish it from the Broeck region. Pfenning and Ford discussed the warp region in [4].

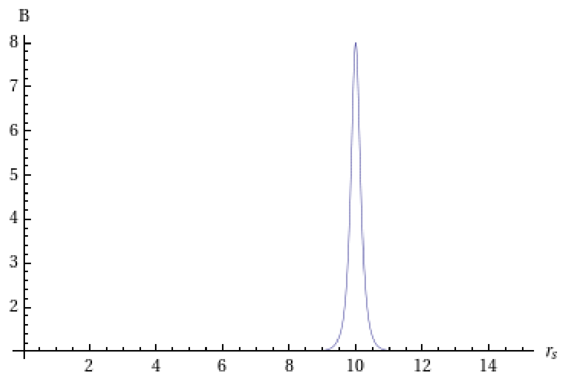


Figure 13: The peak function B for $\sigma = 3, P = 3, D = 10$.

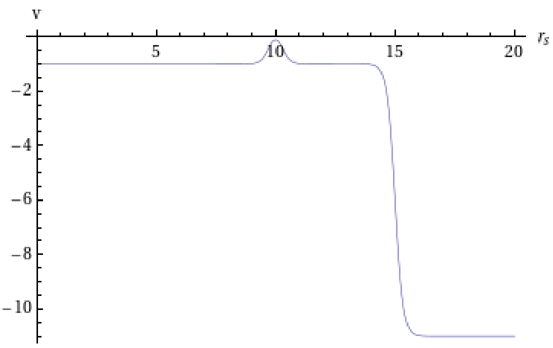


Figure 14: The velocity v of incoming photons for $D = 10, R = 15, v_s = 10, \sigma = P = 3$

Broeck region using Hart's weight function $B(r_s)$ seems to decelerate and then accelerate the incoming objects again ($r_s = 10$ in figure 14).

At least the latter problem can be solved using the alternative weight function $B(r_s)$ from equation (8.2). The velocity of incoming objects using this alternative weight function is plotted in figure 15.

By choosing D relatively close to the ship, photons or incoming particles can be slowed down in the vicinity of the ship, reducing the danger of collisions.

But this would mean, using Hart's function B, that the outer part of the ship would be disturbed by curvature, if one places D so close to the starship that the objects are decelerated, but not yet accelerated again. Using the alternative weight function B from equation (8.2), this problem does not arise.

According to Hart, pieces of matter too small to be disrupted by the tidal forces will be slowed down in the Broeck region just like the photons. For larger pieces of matter, they will become tidally disrupted by the Broeck regions and deflected [5, p. 7].

If Hart is right, this should make interstellar warp flights much more safe.

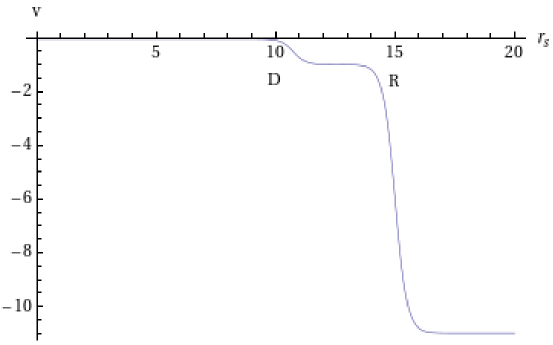


Figure 15: The velocity v of incoming photons for $D = 10, R = 15, v_s = 10, \sigma = P = 3$ and an alternative weight function $B(r_s)$ as given in equation (8.2).

8.5 The Horizon Problem

One other importaint obstacle against the warp drive is the so-called horizon problem: it states that at superluminal velocities, the warp bubble becomes causally disconnected from the starship inside the warp bubble [6].

Loup et al. have shown in their paper *A causally connected superluminal Warp Drive spacetime* [6] that the region of the warp bubble that is required to control the bubble, is still connected to the starship.

Furthermore, Hart et al. pointed out [5, p. 5] that using Broecks's enhancement of the warp metric, the ship will be able to send information in front of the warp bubble:

Hart shows that photons being sent out forward from the ship will leave the warped space, reach the external spacetime and can be detected by an observer far in front of the ship.

The observer on the ship, on the other side, loses contact with the photons in a part of the Pfenning region. This behaviour is similar to the event horizons of black holes, in which a remote observer never sees the photons crossing the event horizons but an observer inside the hole would see the photons go into the singularity [5, p. 6].

But the important result is that it is possible to send information from the ship in front of the warp bubble and, therefore, the horizon problem, according to Hart, can be regarded as solved.

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9 Conclusion

As we have seen, the basic ideas of the warp drive are competitively simple. But without further corrections, the warp drive in its simplest form causes many problems — we have only discussed a few of them.

But the topic appears still to be very active. Currently, there are 67 papers regarding the warp drive on `http://arxiv.org` and many of them try to solve possible problems.

Therefore, even if major difficulties remain, one should not give up on the warp drive, yet. According to *Star Trek: The First Contact*, the first warp flight will take place in 2063. Considering the rapid progress in science and engineering, we might still have a chance to keep this term — perhaps a small chance. But the stimulus to drive the development of a warp drive forward is still there: the wish to be finally able to visit the thousands of stars of our night sky.

10 Bibliography

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Hyperlinks The bibliography is provided as a PDF on the enclosed CD ROM. In this PDF, you can directly click on the hyperlinks instead of typing in the urls.

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